mately 1.5. If the lowest value of $|\Gamma_L|$ is sufficiently small, the reflection from the thin dielectric tube may be equal to, or larger than, this value and no bead is required to give a matched condition. The reflection from the thin dielectric tube can be calculated by first calculating the equivalent dielectric constant of the medium in the transmission line where the tube is located and then by computing $|\Gamma_1|$ from the equation

$$|\Gamma_1| = \frac{\sqrt{\epsilon} - 1}{\sqrt{\epsilon} + 1}. \qquad (5)$$

In Fig. 5, the quantity $|\Gamma_1|$ is plotted as a function of dielectric constant, over a realistic range of values. The use of a bead that only partially fills the coaxial line (or no bead at all) is desireable because it increases the stability of the instrument at very low values of reflection coefficient.

One significant result that has been obtained in using the very low-reflection sliding terminations is the capability to tune the reflectometer down to the point where the nonuniformities of the precision coaxial line become the limiting factor in the tuning operation, and hence the limiting factor in obtainable accuracy of a reflection coefficient measurement. In this light, the sliding termination should be very useful in evaluating the uniformity of coaxial lines as well as for improving the present obtainable accuracy of coaxial reflectometer measurements.

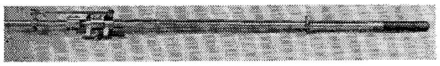

Fig. 3—$\frac{3}{4}$-inch adjustable sliding termination.

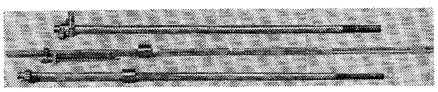

Fig. 4—$\frac{9}{16}$-inch adjustable sliding termination.

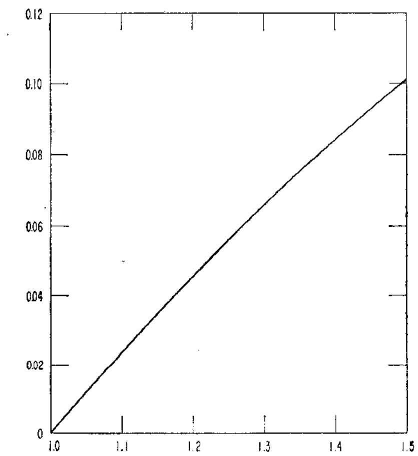

Fig. 5—Graph of (5).

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Approximate Method of Determining the Cutoff Frequencies of Waveguides of Arbitrary Cross Section

Theory

Electromagnetic propagation in a long, prismatic waveguide obeys the scalar Helmholtz equation

$$-\nabla^2\psi = K^2\psi$$

where

ψ = a potential function
K = frequency parameter
$-\nabla^2$ = positive definite plane Laplacian operator.

If the boundary is a curve natural to any of the common coordinate systems for which Helmholtz equation separates, the solutions can be derived by classical methods and may be expressed in terms of known transcendental functions. For the waveguides having more complicated cross sections, however, the cutoff frequencies can only be approximately determined. There are, however, some technical advantages in using these more complicated cross sections. In circular waveguides because of the axisymmetrical field configurations, the waves do not have directional stability but tend to shift in phase intermittently, producing fading and other undesirable consequences. To minimize this effect one or more logitudinal short vanes are sometimes installed on the wall to "lock" the modes in prescribed directions. These vanes change the cutoff frequencies in the waveguide by an appreciable amount. This correspondence shows that it is advantageous to conformally transform the complicated cross section onto a simpler one, (*i.e.*, the unit circle) where the boundary conditions can be easily satisfied. Transformation functions for many common curves are available in standard references.[1,2] For other curves, approximate transformation function can be determined by the series method.[2] Once the transformation function is known, the problem is reduced to the solution of the transformed equation as follows:

$$-\nabla^2\psi = \left|\frac{dz}{d\xi}\right|^2 K^2\psi \qquad (1)$$

where $z = f(\zeta)$: the transformation function. Many methods are available to solve the above equation approximately. Among them, the collocation method is perhaps the simplest. If greater accuracy of the approximate frequency parameter is desired, one may use the iteration technique suggested by Temple.[3] The first iteration can be expressed in terms of upper and lower bounds as follows:

$$\frac{\int_A u\cdot\nabla^2 u\,dA}{\int_A u^2 dA} - \frac{\dfrac{\int_A v^2 dA}{\int_A u^2 dA} - \left[\dfrac{\int_A u\nabla^2 u\,dA}{\int_A u^2 dA}\right]}{(K_2)^2 - \dfrac{\int_A u\nabla^2 u\,dA}{\int_A u^2 dA}} \le (K_1)^2 \le \frac{\int_A u\nabla^2 u\,dA}{\int_A u^2 dA} \qquad (2)$$

where u and v are two functions which satisfy the boundary conditions and are related by

$$-\nabla^2 u = v. \qquad (3)$$

K_2 is the estimated frequency parameter of the first harmonic and K_1 is the parameter which determines the lowest frequency cutoff point. In this discussion, only the case of the TM waves ($\psi = 0$ at the boundary) is considered. The case of the TE waves merely requires a straight forward modification of the method.

Applications

A. *Rectangular Waveguide*

This case is treated here in order to illustrate the method. The transformation function which maps a square region whose sides are 2a by 2a onto a unit circle is

$$z = A\cdot a\int_0^{\xi}(1+\xi^4)^{-1/2}d\xi;\ A = 1.08 \text{ and } \xi = re^{i\theta}.$$

The solution of the transformed partial differential equation can be expressed by means of a complete set of eigenfunctions

$$\psi(r,\theta) = Re\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{nm}J_n(k_{nm})e^{in\theta}$$

where k_{nm} satisfies the boundary condition

$$J_n(k_{nm}r)\big|_{r=1} = 0.$$

As a first approximation we take

$$\psi(r,\theta) \sim W(\mathrm{r}) = \sum_{m=0} B_{0m}J_0(K_{0m}r).$$

Substitution of the above function into (1) gives the error distribution,

$$\epsilon(r,\theta) = \sum_{m=1}^{N} B_{0m}J_0(k_{0m}r)$$
$$\cdot\left[k_{0m}^2(1 + 2r^4\cos 4\theta + r^8)^{1/2} + (1.08)^2a^2K^2\right].$$

We observe that $\epsilon(r,\theta)$ varies periodically with θ. For simplicity we assume that the mean error does not differ much from $(r, \pi/8)$. We arbitrarily choose five points. The computed first two frequency parameters are

$$K_1 = \frac{2.236}{a} \qquad K_2 = \frac{5.178}{a}$$

which compare favorably with the exact values from the closed form solutions which are

$$K_1 = \frac{2.2214}{a} \qquad K_2 = \frac{4.9673}{a}.$$

Manuscript received July 12, 1963; revised September 23, 1963.

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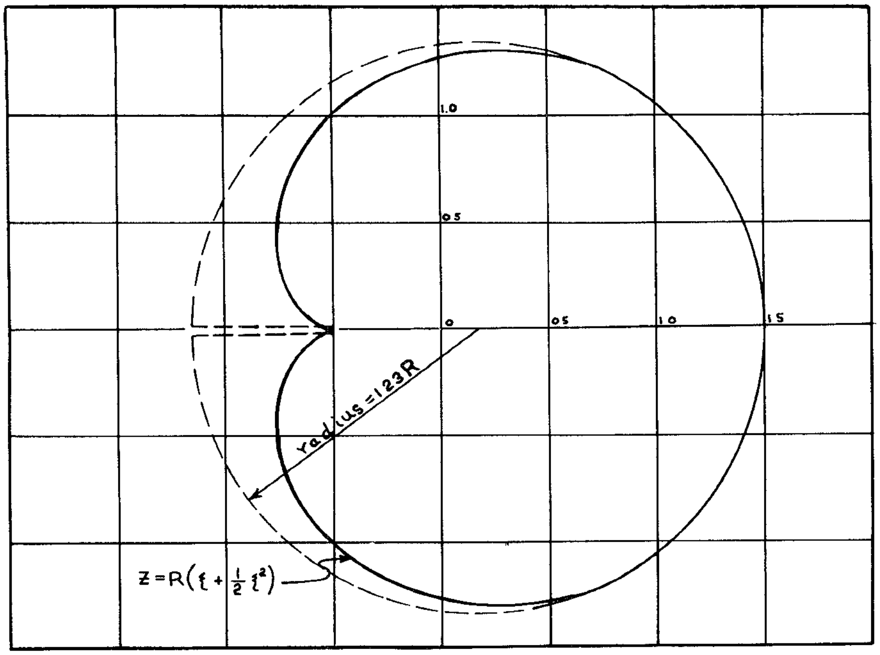

Fig. 1—Approximation of one-vaned waveguide by cardioid.

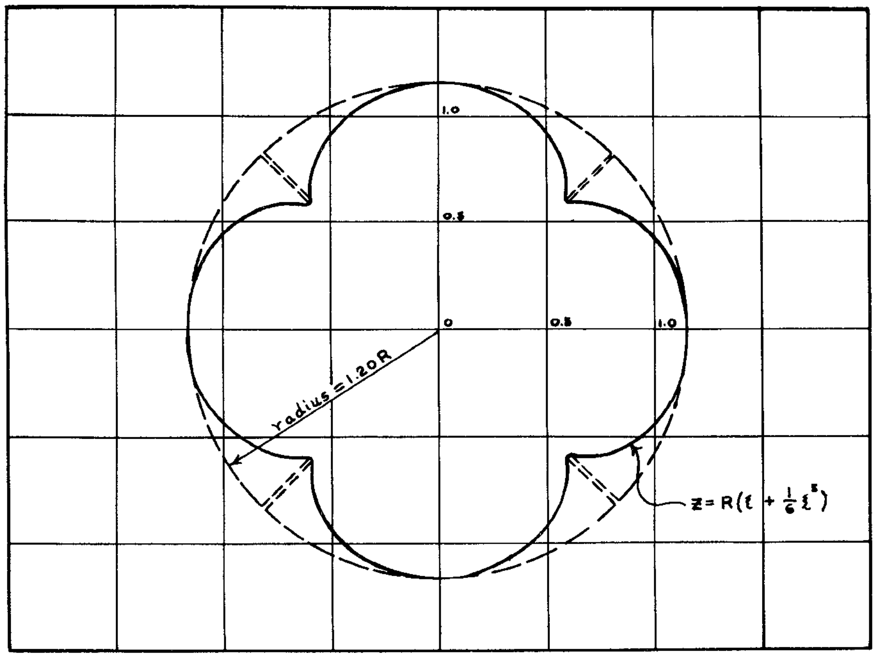

Fig. 2—Approximation of four-vaned waveguide by epitrochoid.

B. Circular Waveguide with One Vane

The transformation function for this configuration is difficult to determine. As a first approximation we represent the shape by a cardioid. The comparison of the wave guide and the mathematical model is shown in Fig. 1. The transformation function which maps a cardioid onto a unit circle is

$$z = R(\xi + \tfrac{1}{2}\xi^2).$$

By a procedure similar to that used in Case A we find

$$K_1 = \frac{2.163}{R}; \qquad K_2 = \frac{4.681}{R}.$$

Based on Case A, we can infer that the results may be a few per cent too high. It is interesting to note that the fundamental frequency of the circular guide without vanes (radius 1.23R) is about $1.96/R$. Comparing the corresponding results, we can conclude that the fundamental frequencies of a circular unvaned guide and vaned guide may differ as much as 10 per cent.

C. Circular Waveguides with Four Diaphragms

The shape of four-vaned circular guides may be represented by an epitrochoid (Fig. 2). The transformation function is

$$z = R(\xi + \tfrac{1}{6}\xi^5).$$

By a procedure similar to the tone used in Case A, we obtain the following frequency parameters:

$$K_1 = \frac{2.388}{R} \qquad K_2 = \frac{5.382}{R}.$$

In this case we have also used the bounding technique (2) to obtain a more accurate answer. After the necessary integration we obtain

$$\frac{2.384}{R} \leq K_1 \leq \frac{2.385}{R}.$$

The exact frequency parameter can be estimated with great certainty. A comparison can be made with an unvaned waveguide whose radius is about 1.2. The vaned waveguide whose cross section is representable by an epitrochoid has a fundamental frequency parameter about 20 per cent higher than that of the corresponding unvaned waveguide.

Concluding Remarks

The method described in this correspondence appears to be suitable to estimate the frequency parameters of any arbitrarily shaped waveguide provided that the transformation function needed to map the section into a circular region can be obtained.

Acknowledgment

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Design Problems and Performance of Millimeter-Wave Fabry-Perot Reflector Plates

The parallel plate Fabry-Perot interferometer and its application have been described by several authors.[1–5] It is par-

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[1] W. Culshaw, "Reflectors for a microwave Fabry-Perot interferometer," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 221–228; April, 1959.

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